Probability distribution functions of derivatives and increments for decaying Burgers turbulence

J. Bec and U. Frisch
Observatoire de la Côte d'Azur, CNRS UMR No. 6529, Boîte Postale 4229, 06304 Nice Cedex 4, France
(Received 4 June 1999)

A Lagrangian method is used to show that the power law with a $-7/2$ exponent in the negative tail of the probability distribution function (PDF) of the velocity gradient and of velocity increments, predicted by E *et al.* [Phys. Rev. Lett. **78**, 1904 (1997)] for forced Burgers turbulence, is also present in the unforced case. The theory is extended to the second-order space derivative whose PDF has power-law tails with exponent -2 at both large positive and negative values and to the time derivatives. PDF's of space and time derivatives have the same (asymptotic) functional forms. This is interpreted in terms of a random Taylor hypothesis.

PACS number(s): 47.27.Gs, 02.50.Ey, 05.60.$-$k

I. INTRODUCTION

E *et al.* [1] made various predictions concerning the one-dimensional Burgers equation

$$\partial_t u + u\partial_x u = \nu \partial_x^2 u + f, \tag{1}$$

with viscosity ν and a random δ-function correlated in time force $f(x,t)$, which is homogeneous, periodic and smooth in the space variable. One prediction concerns the probability density function (PDF) of the velocity gradient $\xi = \partial_x u$. According to Ref. [1], in the limit $\nu \to 0$, the statistically stationary solution of Eq. (1) has a PDF

$$p(\xi) \propto |\xi|^{-7/2} \quad \text{for } \xi \to -\infty. \tag{2}$$

This power-law range is due to preshocks, nascent shocks with a cubic root structure, as discussed by Fournier and Frisch [2]. There has been an interesting controversy about this negative tail of the PDF, which we shall not try to summarize here (see, e.g., Refs. [3–5] and references therein). There is no complete proof at this moment of the validity of the $-7/2$ law, but significant progress has been made recently [6]. We shall not dwell now on the issue of the validity of the $-7/2$ law for forced Burgers turbulence.

It is our intention here to show that the $-7/2$ law is also present in unforced decaying Burgers turbolence. Specifically, we shall consider solutions of Eq. (1) in the limit $\nu \to 0$ with $f=0$ and random zero-mean-value initial conditions $u_0(x)$ which are periodic (a unit period is assumed for convenience), statistically homogeneous, and sufficiently smooth. One such instance is to take Gaussian initial conditions with a spectrum that decreases exponentially at high wave numbers. Such ''large-scale'' initial conditions will develop nonsmooth features (preshocks and shocks) after some (random) time.

This paper is organized as follows. In Sec. II we consider the deterministic problem in Lagrangian coordinates and identify the preshock events leading to large negative gradients. In Sec. III we derive the $-7/2$ law for the PDF of the first derivative. In Sec. IV, we derive similar laws for higher-order space derivatives and the time derivative. In Sec. V we derive the corresponding results for the PDF of space increments. In Sec. VI we make concluding remarks.

II. THE LAGRANGIAN REPRESENTATION AND PRESHOCKS

In the absence of force and of viscous dissipation and as long as no shock has appeared, the Burgers equation (1) has the obvious solution

$$u(x,t) = u_0(a), \quad a = L_t^{-1}x, \tag{3}$$

where

$$L_t : a \mapsto a + t u_0(a), \tag{4}$$

is called the *naive Lagrangian map*. This is indeed just a statement that the velocity of a fluid particle is conserved in Lagrangian coordinates. (Following standard tradition, we denote Lagrangian initial coordinates by a and Eulerian coordinates by x.)

A remarkable property of the unforced Burgers equation in the limit of zero viscosity, which follows from the Hopf-Cole solution (see Refs. [7,8] for details), is that Eq. (3) remains valid in the presence of shocks provided the naive Lagrangian map is replaced by the (proper) Lagrangian map $\mathcal{L}_t$. The latter is defined as follows. First, we define the initial potential (up to an additive constant) by

$$u_0(a) = -\partial_a \psi_0(a). \tag{5}$$

We then define the Lagrangian potential by

$$\varphi(a,t) \equiv -\frac{a^2}{2} + t\psi_0(a) \tag{6}$$

and observe that the naive Lagrangian map is simply the negative gradient of the Lagrangian potential:

$$L_t a = -\frac{\partial}{\partial a}\varphi(a,t). \tag{7}$$

The Lagrangian map is defined as

$$\mathcal{L}_t a \equiv -\frac{\partial}{\partial a}\varphi_c(a,t), \tag{8}$$

where $\varphi_c(a,t)$ is the *convex hull* with respect to a of the Lagrangian potential $\varphi(a,t)$. The convex hull of a function

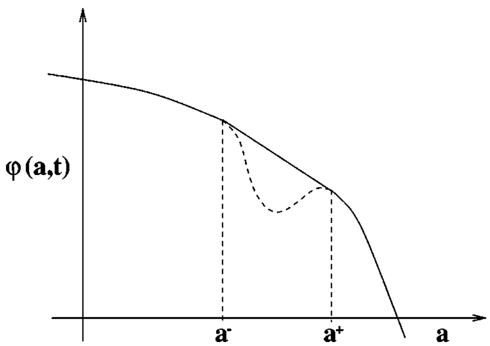

FIG. 1. Lagrangian potential and its convex hull in the presence of a shock interval extending from a^- to a^+.

$f(a)$ can be defined as the smallest piecewise differentiable function that is greater than or equal to $f(a)$ for all a and such that its derivative is nonincreasing.

The graph of the convex hull of $\varphi(a,t)$ is made of pieces of the graph of the function $\varphi(a,t)$ joined by linear segments, sitting over the Lagrangian shock intervals, as shown in Fig. 1. Hence, the Lagrangian map coincides with the naive Lagrangian map except over the Lagrangian shock intervals where it is constant (see Fig. 2). Thus, Eq. (3) with L_t given by (4) remains valid outside the Lagrangian shock intervals.

We can now use this solution (3),(4) to calculate the Eulerian velocity gradient, i.e., its first-order space derivative, in Lagrangian coordinates. Differentiating Eq. (3) and using Eq. (4), we obtain

$$\partial_x u(x,t)=u_0'(a)\frac{1}{\partial_a x}=\frac{u_0'(a)}{1+tu_0'(a)}, \tag{9}$$

where $u_0'(a)\equiv du_0(a)/da$. We immediately observe that, for $t>0$, the only way in which this gradient can become large and negative is to have a very small denominator in Eq. (9). For the kind of smooth initial conditions considered here, the denominator is necessarily positive for sufficiently small times. Let a_* be the location where the initial velocity gradient $u_0'(a)$ achieves its minimum over the period. At

$$t=t_*=\min_a\left[-\frac{1}{u_0'(a)}\right], \tag{10}$$

we have the first preshock, i.e., a shock is born [2]. Subsequently, other (less negative) local minima of $u_0'(a)$ may also produce preshocks, provided that the corresponding location has not already been captured by a previously generated ''mature shock.'' Thus, large negative Eulerian gradients must come from the neighborhood of preshocks.

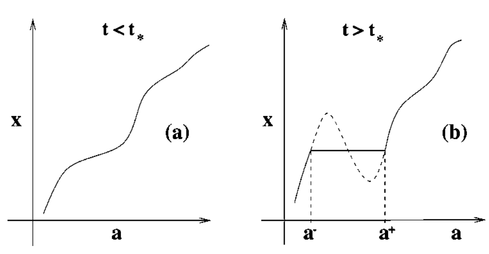

FIG. 2. Lagrangian map before (a) and after (b) the appearance of a shock. The naive Lagrangian map is shown as a dashed line.

We now recall, for use in later sections, the local (normal) form of the Lagrangian and Eulerian solutions near a preshock. Let a_* be a local negative minimum of $u_0'(a)$. We then have (generically)

$$u_0'(a_*)<0, \quad u_0''(a_*)=0, \quad u_0'''(a_*)>0. \tag{11}$$

Taylor expanding near a_*, we have

$$u_0(a)\simeq u_0(a_*)+u_0'(a_*)(a-a_*)+\frac{u_0'''(a_*)}{6}(a-a_*)^3. \tag{12}$$

By Eq. (4), for t near $t_*=-1/u_0'(a_*)$, the naive Lagrangian map is given by

$$x\simeq a_*+tu_0(a_*)+\frac{t_*-t}{t_*}(a-a_*)+\frac{t_*u_0'''(a_*)}{6}(a-a_*)^3. \tag{13}$$

Hence, for given x and t, near $x(a_*,t_*)$ and t_*, respectively, the naive Lagrangian map can be inverted by solving (to leading order) a cubic equation. For $t\leqslant t_*$, this equation has a single real solution and the naive Lagrangian map coincides with the Lagrangian map. For $t=t_*$, the time of the preshock, the equation simplifies and its solution reads

$$a-a_*\simeq\left[\frac{6}{t_*u_0'''(a_*)}[x-x(a_*,t_*)]\right]^{1/3}. \tag{14}$$

Substitution in Eq. (12) gives

$$u(x,t_*)\simeq u_0(a_*)-\frac{1}{t_*}\left[\frac{6}{t_*u_0'''(a_*)}[x-x(a_*,t_*)]\right]^{1/3}. \tag{15}$$

This is the well known Eulerian cubic root structure of preshocks. For t slightly in excess of t_*, the naive Lagrangian map is not monotonic and cannot be inverted (otherwise there would be three branches). There is now a shock. The corresponding shock interval can be determined by using the convex hull construction on the Lagrangian potential. To leading oder, it is found that the shock interval extends from a^- to a^+, which are such that $x(a^\pm,t)=a_*+tu_0(a_*)$, namely,

$$a^\pm-a_*\simeq\pm\left[\frac{6(t-t_*)}{t_*^2u_0'''(a_*)}\right]^{1/2}. \tag{16}$$

This condition expresses that, to leading order, the Eulerian location of the shock remains fixed in a frame moving with the velocity $u_0(a_*)$.

In Fig. 3, we have illustrated the Eulerian structure of the solution at three times: just before, at, and just after the time t_* of the preshock.

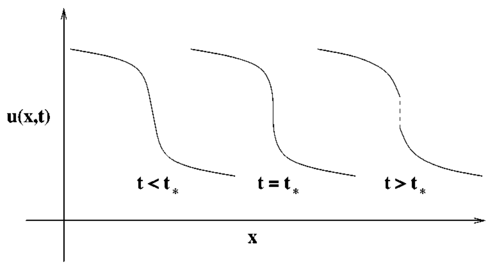

FIG. 3. Eulerian structure of the solution (a) just before a preshock; (b) at the time of a preshock; (c) just after a preshock.

III. THE PDF OF THE VELOCITY GRADIENT

Our purpose is to derive the behavior for $\xi \to -\infty$ of the PDF of the velocity gradient

$$p(\xi;x,t) \equiv \langle \delta(\xi - \partial_x u(x,t)) \rangle, \tag{17}$$

where angular brackets denote ensemble averaging over the random initial condition u_0. By the assumed homogeneity, $p(\xi;x,t)$ is obviously independent of x [and will subsequently be denoted $p(\xi,t)$]. It follows that

$$p(\xi,t) = \left\langle \int_0^1 \delta(\xi - \partial_x u(x,t)) dx \right\rangle. \tag{18}$$

Having thus a representation of our PDF as a space integral over the Eulerian coordinate x, we can make the change of variable from Eulerian to Lagrangian coordinates, using the map $\mathcal{L}_t$. The same idea was used in Ref. [2] to calculate the Fourier transform of the Eulerian solution during the early phase of regularity (before the appearance of shocks). This idea also works for later times provided we use the Lagrangian map, which differs from the naive Lagrangian map (4) only by the exclusion from the basic periodicity interval $[0,1[$ of the Lagrangian shock intervals. Let us denote by $D_L(t)$ the set of so-called regular points, i.e., Lagrangian points that do not belong to shock intervals. Using the Lagrangian representation (9) of the velocity gradient, we obtain from Eq. (18)

$$p(\xi,t) = \left\langle \int_{D_L(t)} \delta\left(\xi - \frac{u_0'(a)}{1 + t u_0'(a)} \right) |1 + t u_0'(a)| da \right\rangle. \tag{19}$$

Note that $1 + t u_0'(a)$ is the Jacobian of the Lagrangian map. Hereafter we shall use several times the formula

$$\delta(f(y)) = \sum_j \frac{1}{|f'(y_j)|} \delta(y - y_j), \tag{20}$$

where the y_j's are the zeros of f and f is assumed to be sufficiently smooth.

Let us denote by b_k the (discrete) Lagrangian locations where the arguments of the delta function in Eq. (19) vanish, i.e., locations which are the roots of

$$\xi - \frac{u_0'(b)}{1 + t u_0'(b)} = 0. \tag{21}$$

Using Eq. (20) and evaluating the derivative of the argument of the delta function in Eq. (19) at the point where this argument vanishes, we can rewrite the PDF of the gradient as

$$p(\xi,t) = \frac{1}{|1 - t\xi|^3} \sum_k \left\langle \frac{1}{|u_0''(b_k)|} \int_{D_L(t)} \delta(a - b_k) da \right\rangle, \tag{22}$$

where the integral over the delta function may be viewed as shorthand for the indicator function of $D_L(t)$ [equal to one if $b_k \in D_L(t)$ and to zero otherwise]. Note that the right-hand side (rhs) of Eq. (22) has a $|\xi|^{-3}$ dependence for large ξ if we take into account only the first factor. Actually, we shall see that the presence of $u_0''(b_k)$ in the denominator provides an additional $|\xi|^{1/2}$ factor and that realizability conditions provide a $|\xi|^{-1}$ factor, so that the PDF will be proportional to $|\xi|^{-7/2}$.

So far, we have not made any expansion. Let us now concentrate on the case of large negative ξ's. As observed in Sec. II, this happens only in the neighborhood of preshocks. The latter originate from Lagrangian locations at which $u_0(a)$ has an inflection point satisfying Eq. (11). Let a_{*j} be the discrete set of such locations.

We show now by perturbation theory that, for each such point, there are zero or two roots of Eq. (21). Indeed, using the Taylor expansion (12), in Eq. (21), we obtain

$$(b - a_{*j})^2 \simeq \frac{2}{t u_0'''(a_{*j})} \left[-\frac{1}{t\xi} - 1 - t u_0'(a_{*j}) \right], \tag{23}$$

which has either two roots (denoted $b_j^{\pm}$) or none, depending on the sign of the rhs Defining $t_{*j} \equiv -1/u_0'(a_{*j})$, it is now convenient to distinguish the cases $t \leqslant t_{*j}$ and $t > t_{*j}$, corresponding respectively to before and after the preshock. Before, the Lagrangian shock interval near a_{*j} is empty; the two conditions that $t \leqslant t_{*j}$ and that the rhs of Eq. (23) is positive read

$$-\frac{1}{t} \leqslant u_0'(a_{*j}) < -\frac{1}{t} - \frac{1}{t^2 \xi}. \tag{24}$$

After t_{*j}, the Lagrangian shock interval is defined by Eq. (16). Since shock intervals are excluded from the integral (22), acceptable solutions must be outside of such intervals. This and $t > t_{*j}$ again provide two conditions, namely,

$$-\frac{1}{t} + \frac{1}{2t^2\xi} < u_0'(a_{*j}) < -\frac{1}{t}. \tag{25}$$

We observe now that the two conditions (24) and (25) may be written as a single condition

$$-\frac{1}{t} + \frac{1}{2t^2\xi} < u_0'(a_{*j}) < -\frac{1}{t} - \frac{1}{t^2\xi}, \tag{26}$$

which, for large negative ξ, restricts $u_0'(a_{*j})$ to being near $-1/t$ in a small interval of length $-3/(2t^2\xi)$. We shall denote by $\mathbb{1}_{PS}(u_0'(a_{*j});t,\xi)$ the indicator function equal to one if $u_0'(a_{*j})$ is in this preshock realizability interval and to zero otherwise.

The integral over a appearing in Eq. (22) is over the complement of shock intervals. The above analysis only takes care of those nascent shock intervals that can be calculated perturbatively. A further condition is that the point a_{*j} should not be within a mature shock interval that was created before time t. Because of the convex hull construction, this is a global geometrical constraint that cannot, in general, be expressed perturbatively. We shall denote by $\mathbb{1}_{D_G(t)}(a)$ the indicator function equal to one if a is outside such a global shock interval and to zero otherwise.

We now return to Eq. (22). Since $b_k = b_j^{\pm}$, which is close to the a_{*j}'s where u_0'' vanishes, we can use the Taylor expansion $|u_0''(b_j^{\pm})| \simeq |b_j^{\pm} - a_{*j}| u_0'''(a_{*j})$, which by Eq. (23) takes the same value for b_j^+ and b_j^-. Hence, the contribution to the integral in Eq. (22) of the two b_k points in the neighborhood of each point a_{*j} is $2\mathbb{1}_{PS}(u_0'(a_{*j});t,\xi)\mathbb{1}_{D_G(t)}(a_{*j})$, where the products of the indicator functions express the shock interval exclusion. Using Eq. (23), we now obtain

$$p(\xi,t) \simeq \frac{t^{1/2}}{|t\xi|^3} \times \left\langle \sum_j \frac{2\mathbb{1}_{PS}(u_0'(a_{*j});t,\xi)\mathbb{1}_{D_G(t)}(a_{*j})}{\left\{2u_0'''(a_{*j})\left[-\frac{1}{t\xi} - 1 - tu_0'(a_{*j})\right]\right\}^{1/2}} \right\rangle. \tag{27}$$

Since the sum is over points of vanishing u_0'' with $u_0''' > 0$, by use of Eq. (20) this may be rewritten as

$$p(\xi,t) \simeq \frac{(2t)^{1/2}}{|t\xi|^3} \left\langle \int_0^1 da \frac{(u_0'''(a))^{1/2}\delta(u_0''(a))}{\left[-\frac{1}{t\xi} - 1 - tu_0'(a)\right]^{1/2}} \times \mathrm{H}(u_0'''(a))\mathbb{1}_{PS}(u_0'(a);t,\xi)\mathbb{1}_{D_G(t)}(a) \right\rangle, \tag{28}$$

where $\mathrm{H}(\cdot)$ is the Heaviside function.

Now, interchanging the mean value and the integration over a, we observe that, because of homogeneity, the integrand does not depend on a. Hence, the integration over a can be omitted. Let us denote by $p_{3,0}(u',u'',u'''|\mathbb{1}_{D_G(t)}=1)$ the joint PDF of the first three derivatives of the *initial* velocity at an arbitrary Lagrangian location, knowing that this location is not within a mature shock interval at time t. We can then write

$$p(\xi,t) \simeq \frac{(2t)^{1/2}}{|t\xi|^3} \int_0^\infty du''' \int_{-(1/t)+(1/2t^2\xi)}^{-(1/t)-(1/t^2\xi)} du' \times \frac{(u''')^{1/2} p_{3,0}(u',0,u'''|\mathbb{1}_{D_G(t)}=1)}{\left[-\frac{1}{t\xi} - 1 - tu'\right]^{1/2}}. \tag{29}$$

In Eq. (29), the variable u' is constrained to remain very close to $-1/t$ for $\xi \to -\infty$. Assuming that the density $p_{3,0}$ is smooth in its u' argument, we can replace the latter by $-1/t$ and carry out the remaining integration over u', to obtain

$$p(\xi,t) \simeq 2\sqrt{3}t^{-4}D(t)|\xi|^{-7/2}, \quad \xi \to -\infty, \tag{30}$$

where

$$D(t) \equiv \int_0^\infty du'''(u''')^{1/2} p_{3,0}\left(-\frac{1}{t},0,u'''\middle|\mathbb{1}_{D_G(t)}=1\right). \tag{31}$$

This concludes the derivation of the $-7/2$ law for decaying Burgers turbulence. The time-dependent constant $D(t)$ is expressed in terms of a conditional probability that cannot be calculated without solving a global geometric random problem [9]. A more explicit form is obtained for small t: the condition $\mathbb{1}_{D_G(t)}=1$ may then be omitted and the integral (31) can be calculated, e.g., in the Gaussian case. Indeed, for the kind of large-scale initial conditions assumed here, if, near some point a_*, the initial velocity gradient u_0' achieves a very large negative minimum close to $-1/t$, the other minima will be above $-1/t$ with a probability very close to unity, so that it is nearly certain that no mature shocks have been formed.

IV. HIGHER-ORDER SPACE DERIVATIVES AND TIME DERIVATIVES

We begin with the second Eulerian space derivative $\partial_x^2 u$. The method is rather similar to the one used for the first derivative. So, we shall avoid repeating details. From Eqs. (3), (4), and (9), it follows that

$$\partial_x^2 u = \partial_a(\partial_x u)\frac{1}{\partial_a x} = \partial_a\left[\frac{u_0'(a)}{1+tu_0'(a)}\right]\frac{1}{1+tu_0'(a)} = \frac{u_0''(a)}{[1+tu_0'(a)]^3}. \tag{32}$$

Denoting by $p^{(2)}(\xi,t)$ the PDF of $\partial_x^2 u$, we have, as before,

$$p^{(2)}(\xi,t) = \left\langle \int_{D_L(t)} \delta\left(\xi - \frac{u_0''(a)}{[1+tu_0'(a)]^3}\right)|1+tu_0'(a)|da \right\rangle. \tag{33}$$

For Gaussian statistics or, more generally, when the probability of very large values of $u_0''(a)$ is very small, large values of $\partial_x^2 u$ will be due overwhelmingly to small denominators. That is, they will again originate from the neighborhood of preshocks. Near an inflection point a_{*j} of the kind considered in Sec. III, we have

$$\partial_x^2 u \simeq \frac{u_0'''(a_{*j})(a-a_{*j})}{\left[1+tu_0'(a_{*j}) + \frac{t}{2}u_0'''(a_{*j})(a-a_{*j})^2\right]^3}. \tag{34}$$

This is an odd function of $a-a_{*j}$ that can achieve both large positive and large negative values. (Hence, we obtain power-law tails at both ends.)

As before, in Eq. (33) we change from a delta function over ξ to delta functions over those Lagrangian locations b_k where the second space derivative is equal to ξ. It is easily shown that the condition whereby the rhs of Eq. (34) is equal to ξ has either two solutions (on the same side of a_{*j}) or none. Obtaining these solutions explicitly requires solving an algebraic equation of degree six in $b-a_{*j}$. Nevertheless, the conditions for the existence of such solutions can be written explicitly, as before. In the early shock phase there may now be either two, or one, or zero b_k's outside of the shock interval. The length of the realizability interval in the variable $u_0'(a_{*j})$ is now $O(|\xi|^{-2/5})$. The coefficient in front of the distributions $\delta(a-b_k)$ is now found to be $O(|\xi|^{-8/5})$. It follows that

$$p^{(2)}(\xi,t)\propto|\xi|^{-2},\quad \xi\to\pm\infty. \tag{35}$$

The time-dependent constant in front of the -2 power law can again be expressed in terms of the conditional joint probability $p_{3,0}$ already introduced, but this is very cumbersome since it involves the solution of the aforementioned equation of degree six.

The theory can be extended to higher-order space derivatives but becomes even more cumbersome. Somewhat superficial inspection (mostly by dimensional analysis) indicates that the PDF's have in such cases power-law tails with exponent $-(3n+4)/(3n-1)$ [10]. For even n the tail is present for both large negative and large positive values. For odd $n>2$ it is certainly present for large negative values and may also be present for positive ones (e.g., for $n=3$).

Finally, we turn to the Eulerian time derivative. We define

$$p_{\partial_t u}(\eta,t)=\langle\delta(\eta-\partial_t u(x,t))\rangle. \tag{36}$$

From Eq. (3) we have

$$\partial_t u(x,t)=u_0'(a)\partial_t a, \tag{37}$$

where $\partial_t a$ is calculated for a given Eulerian position x. Time differentiation of $x=a+tu_0(a)$ gives

$$\partial_t a=-\frac{u_0(a)}{1+tu_0'(a)}. \tag{38}$$

Hence,

$$\partial_t u(x,t)=-\frac{u_0(a)u_0'(a)}{1+tu_0'(a)}. \tag{39}$$

Note that the rhs is just $-u\partial_x u$, which could have been deduced from the Eulerian inviscid equation. Substituting Eqs. (39) in (36) and proceeding almost exactly as in Sec. III, we obtain

$$p_{\partial_t u}(\eta,t)\simeq 2\sqrt{3}t^{-4}E^{\pm}(t)|\eta|^{-7/2},\quad \eta\to\pm\infty, \tag{40}$$

where

$$E^{\pm}(t)\equiv\pm\int_0^{\pm\infty}du|u|^{5/2}\int_0^{\infty}du'''(u''')^{1/2}\times p_{4,0}\left(u,-\frac{1}{t},0,u'''\middle|\mathbb{1}_{D_G(t)}=1\right), \tag{41}$$

which involves the joint PDF $p_{4,0}(u,u',u'',u'''|\mathbb{1}_{D_G(t)}=1)$ of the initial velocity and its first three derivatives at an arbitrary location, knowing that this location is not within a mature shock interval at time t. Note that the PDF of the time derivative (40) is just the PDF of the space derivative (30) with the change of variable $\eta\to-u_0\xi$ and an extra averaging over u_0. This is the result we expect if, in Eq. (39), we neglect the variation of $u_0(a)$ near a preshock. It is easily shown, when doing the complete asymptotic expansion along the same lines as in Sec. III, that this is indeed the case for the leading-order behavior. This theory can again be extended to PDF's of higher-order time derivatives that follow the same power laws as for space derivatives.

Obtaining for the PDF of the Eulerian time derivative the same law as for the space derivative is not very surprising. In high-Reynolds-number turbulent flows it is well known that when there is a large mean flow, the Eulerian temporal structure is, to leading order, determined by the spatial structure in the reference frame of the mean flow (this is often referred to as the ''Taylor hypothesis,'' but is of course a simple asymptotic result). Furthermore, when there is no mean flow, it is generally believed that the small-scale temporal structure is still determined by the spatial structure, since most of the time dependence comes from the sweeping of small-scale eddies by larger energy-containing eddies that have much larger but random velocities. For the case of Burgers turbulence, the identical functional forms of Eqs. (30) and (40) may be seen as a proof of this ''random Taylor hypothesis.'' Note that it is the sweeping by the random velocities of the shocks [$u_0(a)$ at those locations where $u_0'(a)<0$, $u_0''(a)=0$ and $u_0'''(a)>0$] that determines the interplay of temporal and spatial structures. Since we assumed that the velocity has zero mean value, the random velocities at the shocks can have both signs, so that the $-7/2$ power-law tail appears at both large positive and large negative values of the Eulerian time derivatives. Alternatively, one may calculate the PDF of $\partial_t u$ in the frame moving with the shock (assuming there is a single shock). In this case one obtains a much steeper law $\propto|\eta|^{-6}$. Note that this is not the PDF of the Lagrangian time derivative. For unforced Burgers turbulence in the inviscid limit, this derivative is exactly zero.

V. VELOCITY INCREMENTS

We define the Eulerian velocity increment over a separation Δx as

$$\Delta u_E(\Delta x;x,t)\equiv u(x+\Delta x,t)-u(x,t). \tag{42}$$

Our goal is to find the PDF

$$p_{\Delta u}(\xi,\Delta x,t)\equiv\langle\delta(\xi-\Delta u_E(\Delta x;x,t))\rangle, \tag{43}$$

for values ξ of sign opposite to that of the separation. Let us introduce the Lagrangian velocity increment, defined outside the Lagrangian shock intervals, as

$$\Delta u_L(\Delta x;a,t)\equiv\Delta u_E[\Delta x;a+tu_0(a),t]. \tag{44}$$

Proceeding as at the beginning of Sec. III, we obtain

$$p_{\Delta u}(\xi,\Delta x,t)=\left\langle \int_{D_L(t)}\delta(\xi-\Delta u_L(\Delta x;a,t))|1+tu_0'(a)|da\right\rangle. \tag{45}$$

For a given $t,\Delta x$ and ξ, we must now find those Lagrangian locations, denoted b_k, where the argument of the delta function in Eq. (45) vanishes. For this, it is convenient to associate with each b_k, the point b_k' such that their images by the Lagrangian map $\mathcal{L}_t$ are separated by a distance Δx, while the velocities differ by ξ. We thus have

$$u_0(b_k')-u_0(b_k)=\xi \tag{46}$$

$$b_k'+tu_0(b_k')=b_k+tu_0(b_k)+\Delta x. \tag{47}$$

The equivalent of Eq. (22) is now

$$p_{\Delta u}(\xi,\Delta x,t)=\sum_k \left\langle \left|\frac{[1+tu_0'(b_k')][(1+tu_0'(b_k))}{u_0'(b_k')-u_0'(b_k)}\right| \times\int_{D_L(t)}\delta(a-b_k)da\right\rangle. \tag{48}$$

Here we shall be interested exclusively in situations where

$$|\Delta x|\ll|t\xi|\ll 1, \tag{49}$$

which originate from the neighborhood of preshocks a_{*j} where the Taylor expansion (12) may be used [11]. From Eqs. (46) and (47), we then obtain that b_k is a root of the following quadratic equation in b:

$$(b-a_{*j})^2+t\xi(b-a_{*j})+\frac{t^2\xi^2}{3}+\frac{2}{tu_0'''(a_{*j})}\times\left[1+tu_0'(a_{*j})+\frac{\Delta x}{t\xi}\right]=0. \tag{50}$$

We shall see that Eq. (50), together with the realizability condition of having real roots not belonging to a shock interval, can have zero, one, or two solutions, which we shall denote by b_{jm}. We then approximate the PDF (48) using the Taylor expansion (12) near preshocks to obtain

$$p_{\Delta u}(\xi,\Delta x,t)\simeq\sum_{jm}\left\langle \left|A_{jm}+B_{jm}\right|\int_{D_L(t)}\delta(a-b_{jm})da\right\rangle, \tag{51}$$

where

$$A_{jm}=\frac{[1+tu_0'(a_{*j})+tu_0'''(a_{*j})(b_{jm}-a_{*j})^2/2]^2}{t\xi u_0'''(a_{*j})(b_{jm}-a_{*j})}, \tag{52}$$

$$B_{jm}=t[1+tu_0'(a_{*j})+tu_0'''(a_{*j})(b_{jm}-a_{*j})^2/2]. \tag{53}$$

The realizability conditions associated with Eq. (50) lead to rather involved expressions for the PDF. Simple scaling behavior emerges in two limiting cases. To express the corresponding conditions in reasonably compact fashion, we shall assume that the third derivative $u_0'''(a_{*j})$ is of order unity.

For $|t\xi|\ll|\Delta x|^{1/3}$ the A_{jm} term in Eq. (51) dominates. For $|t\xi|\gg|\Delta x|^{1/3}$, the B_{jm} term dominates. In the former case, A_{jm} can be further approximated:

$$|A_{jm}|\simeq\frac{|\Delta x|^2}{|t\xi|^3|u_0'''(a_{*j})(b_{jm}-a_{*j})|}. \tag{54}$$

(Realizability imposes that ξ and Δx be of opposite sign.) Substitution of Eq. (54) into Eq. (51) leads essentially to the same expression (27) as for the PDF of velocity gradients, provided (i) we replace the Eulerian velocity gradient by $\xi/\Delta x$ and (ii) multiply the PDF (27) by $1/|\Delta x|$. Hence,

$$p_{\Delta u}(\xi,\Delta x,t)\simeq\frac{1}{|\Delta x|}p\left(\frac{\xi}{\Delta x},t\right)\propto|\Delta x|^{5/2}|\xi|^{-7/2} \tag{55}$$

$$\text{for}\quad |\Delta x|\ll|t\xi|\ll|\Delta x|^{1/3},\quad \frac{\xi}{\Delta x}<0. \tag{56}$$

For $1\gg|t\xi|\gg|\Delta x|^{1/3}$ the B_{jm} term in Eq. (51) dominates. Contrary to the former case, the situation is quite different from that which what has been studied in Sec. III. We shall therefore give more detail. With the assumptions made, the condition whereby the quadratic equation (50) has real roots reads

$$u_0'(a_{*j})\leqslant-\frac{1}{t}-\frac{u_0'''(a_{*j})}{24}t^2\xi^2, \tag{57}$$

which, given the positivity of $u_0'''(a_{*j})$, implies that $t>t_{*j}=-1/u_0'(a_{*j})$. It is then easily checked that one of the two roots of Eq. (50) is not acceptable because it is within the Lagrangian shock interval. The condition whereby the other one is outside this interval reads

$$-\frac{1}{t}-\frac{u_0'''(a_{*j})}{6}t^2\xi^2\leqslant u_0'(a_{*j}). \tag{58}$$

Equations (57) and (58) now play the role of the realizability conditions (26) in Sec. III. Proceeding along the same lines as in Sec. III, we obtain

$$p_{\Delta u}(\xi,\Delta x,t)\simeq\frac{t^3}{8}F(t)|\Delta x||\xi|, \tag{59}$$

$$\text{for}\quad |\Delta x|^{1/3}\ll|t\xi|\ll 1,\quad \frac{\xi}{\Delta x}<0, \tag{60}$$

where

$$F(t)\equiv\int_0^\infty du'''(u''')^2\,p_{3,0}\left(-\frac{1}{t},0,u'''\,\middle|\,\mathbb{1}_{D_G(t)}=1\right),\qquad(61)$$

and $p_{3,0}$ is defined as in Sec. III.

We observe that the ξ dependence of our results (55) and (59) for decaying burgers turbolence is essentially the same as that proposed in Ref. [1] for the forced case.

VI. CONCLUDING REMARKS

We have shown here that several results proposed in Ref. [1] for forced Burgers turbulence are also valid for decaying Burgers turbulence and can actually be derived by systematic asymptotic expansions, using a Lagrangian approach. The results that carry over from the forced to the unforced case are those involving preshocks: $-7/2$ power law for the PDF of velocity gradients and increments and $+1$ power law for the PDF of increments over suitable ranges. For the tail of the PDF of gradients at large *positive* ξ, a decaying exponential law of the argument ξ^3 is generally proposed [1,12–15]. This result, unrelated to preshocks, has no counterpart in the decaying case. Indeed, it follows then from $D\xi/Dt=-\xi^2$, where D/Dt denotes the Lagrangian derivative, that the PDF of ξ is exactly zero for $\xi>1/t$.

Our results involving the $-7/2$ power law are quite explicit: for example, we obtain the (time-dependent) constant $D(t)$, given by Eq. (31), before the power law (30). The expression $D(t)$ for short times, when mature shocks have negligible probability, can be written explicitly in terms of the joint PDF of the first three derivatives of the initial velocity at an arbitrary location. For later times we need the conditional PDF, knowing known that no mature shock is present at that location. Obtaining this exactly, say for Gaussian initial conditions, may be very hard. But it is possible to construct lower bounds. For example, large-deviations theory may be used to show that $-\ln D(t)=O(\ln^2 t)$ for $t\to\infty$ [16].

We observe that, formally, our results can easily be extended from the case of a space-periodic homogeneous initial condition $u_0(a)$ (as assumed in Ref. [1]) to that of a random homogeneous mixing initial condition defined on the whole real line. For this it suffices to use ergodicity and to replace, in Eq. (18), the mean of the integral over the period by $\lim_{L\to\infty}[1/(2L)]\int_{-L}^{+L}$. After this, the calculation is essentially unchanged.

It is of interest to point out that without careful handling of the shock conditions, an incorrect $|\xi|^{-3}$ power law is obtained for the left-hand tail of the PDF of gradients. Indeed, the PDF (22) has an overall $|\xi|^{-3}$ factor in front of the rhs (for large $|\xi|$). If in the remaining factor we perform the integration over the whole Lagrangian interval $[0,1[$ without excluding the Lagrangian shock interval, we obtain an order unity contribution. It has already been pointed out in Ref. [17] that a $|\xi|^{-3}$ law is obtained from multivalued solutions of the Riemann equation. This is indeed equivalent to using the naive Lagrangian map; note that it gives the correct answer when using the Zeldovich approximation in cosmology, which allows ''multistream'' solutions [18,19].

It is shown in Ref. [20] that there is a simple relation between the velocity gradient and the density of an advected passive scalar. When this density is initially uniform, this implies that the power law with exponent $-7/2$ also applies to the tail of the density PDF. Analogous results can be obtained in several dimensions, where they have cosmological implications; this requires the study of singularities of multidimensional convex hulls [19].

Finally, there is a problem which is in a way intermediate between the decaying and the forced cases, namely, ''kicked'' Burgers turbulence. The space-periodic force f appearing in Eq. (1) is then of the form

$$f(x,t)=\sum_j f_j(x)\,\delta(t-t_j),\qquad(62)$$

where the $f_j(x)$ are deterministic or random prescribed functions. Between the kicking times t_j we have decaying dynamics. At time t_j the velocity undergoes a temporally (but not spatially) discontinuous change $f_j(x)$. When the t_j's are equally spaced and all the $f_j(x)$'s are equal, the solution of the Burgers equation converges to a space-time-periodic function. PDF's obtained by space and time averaging have exactly the same scaling properties as those obtained here for random decaying Burgers turbulence. Very clean scaling can be obtained by numerically simulating this problem using a modification of the fast Legendre transform method of Refs. [8,21]. Such questions are discussed in Ref. [5].

ACKNOWLEDGMENTS

We thank M. Blank, S. Chen, W. E, S. Gurbatov, K. Khanin, R. Kraichnan, A. Noullez, B. Shraiman, E. Vanden Eijnden, and B. Villone for fruitful discussions.

[1] W. E, K. Khanin, A. Mazel, and Ya.G. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).
[2] J.D. Fournier and U. Frisch, J. Mec. Theor. Appl. (France) **2**, 699 (1983).
[3] R.H. Kraichnan, Phys. Fluids (to be published); e-print chao-dyn/9901023.
[4] W. E and E. Vanden Eijnden, Phys. Rev. Lett. **83**, 2572 (1999),
[5] J. Bec, U. Frisch, and K. Khanin, e-print chao-dyn/9910001.
[6] W. E and E. Vanden Eijnden, Commun. Pure Appl. Math. (to be published; e-print chao-dyn/9904028.
[7] Ya.G. Sinai, Commun. Math. Phys. **148**, 601 (1992).
[8] M. Vergassola, B. Dubrulle, U. Frisch, and A. Noullez, Astron. Astrophys. **289**, 325 (1994).
[9] Nevertheless, it can be shown, using a Chebyshev inequality, that $D(t)>0$ (M. Blank, private communication, 1999).
[10] This result has also been obtained by K. Khanin (private communication, 1999).
[11] We are thereby limiting ourselves to the consideration of small-amplitude shocks which are newborn. Small shock amplitudes can also be obtained when a shock has decayed for a time long compared to its time of creation; for times order

unity, the latter phenomenon happens only with a very small probability and does not contribute to the leading-order expression of the PDF of velocity increments over small separations.

[12] A.M. Polyakov, Phys. Rev. E **52**, 6183 (1995).

[13] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).

[14] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **78**, 1452 (1997).

[15] T. Gotoh and R.H. Kraichnan, Phys. Fluids **10**, 2859 (1998).

[16] G. Molchan (unpublished).

[17] W. E and E. Vanden Eijnden, Phys. Fluids **11**, 2149 (1999).

[18] Ya.B. Zeldovich, Astron. Astrophys. **5**, 84 (1970).

[19] U. Frisch, J. Bec, and B. Villone, e-print cond-mat/9912110.

[20] S.N. Gurbatov, A.N. Malakhov, and A.I. Saichev, *Nonlinear Random Waves and Turbulence in Nondispersive Media: Waves, Rays, Particles* (Manchester University Press, Manchester, 1991).

[21] A. Noullez and M. Vergassola, J. Sci. Comput. **9**, 259 (1994).